

Chebyshev Centers That Are Not Farthest Points

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ABSTRACT. In this paper we address the question whether in a given Banach space, a Chebyshev center of a nonempty bounded subset can be a farthest point of the set. Our exploration reveals that the answer depends on the convexity properties of the Banach space. We obtain a characterization of two-dimensional real strictly convex spaces in terms of Chebyshev center not contributing to the set of farthest points. We explore the scenario in uniformly convex Banach spaces and further study the roles played by centerability and M-compactness in the scheme of things to obtain a step by step characterization of strictly convex Banach spaces. We also illustrate with examples the optimality of our results.

1. INTRODUCTION.

In this paper letter X denotes a Banach space, $B_X = \{x \in X : \|x\| \leq 1\}$ and $S_X = \{x \in X : \|x\| = 1\}$ denote the unit ball and the unit sphere of X respectively; $B[x, r] = \{y \in X : \|x - y\| \leq r\}$ is the closed ball with center x and radius r and $S[x, r] = \{y \in X : \|x - y\| = r\}$ is the closed sphere with center x and radius r . For a set A , $|A|$ denotes the cardinality of A , if A is finite then $|A|$ is the number of elements in A . We call a subset A of X *nontrivial* if $|A| \geq 2$. For a nonempty bounded set $A \subseteq X$, its *diameter* is

$$\text{diam}(A) = \sup_{a_1, a_2 \in A} \|a_1 - a_2\|.$$

2010 *Mathematics Subject Classification.* Primary 46B20, Secondary 41A65; 52A21; 47H10.

Key words and phrases. Chebyshev center; farthest point; strict convexity; uniform convexity.

The first author lovingly acknowledges the constant support and encouragement coming from his beloved younger brother Mr. Debdoot Sain. The research of the second author is done in frames of Ukrainian Ministry of Science and Education Research Program 0115U000481; partially it was done during his stay in Murcia (Spain) under the support of MINECO/FEDER project MTM2014-57838-C2-1-P and Fundación Séneca, Región de Murcia grant 19368/PI/14, and partially during his visit to the University of Granada which was supported by MINECO/FEDER project MTM2015-65020-P. The fourth author would like to thank DST, Govt. of India, for the financial support in the form of doctoral fellowship.

The *outer radius* of $A \subseteq X$ at an element $x \in X$ is defined as

$$r(x, A) = \sup_{a \in A} \|x - a\|.$$

The supremum in the definition of $r(x, A)$ may be or may be not attained at some point of A . Let

$$F(x, A) = \{a \in A : \|x - a\| = r(x, A)\}$$

denote the collection of all elements in A which are farthest from $x \in X$. If for an element $x \in X$, $r(x, A)$ is not attained then $F(x, A) = \emptyset$. The collection of all elements in A at which $r(x, A)$ is attained for some $x \in X$ is denoted by $\text{Far } A$ i.e.,

$$\text{Far } A = \bigcup_{x \in X} F(x, A).$$

Recall that the most intriguing unsolved problem about farthest points [6] is whether there exists a nontrivial bounded convex closed subset A of a Hilbert space H with the property that $|F(x, A)| = 1$ for every $x \in H$ (see also [1] and [7]).

The *Chebyshev radius* $r(A)$ of A is given by $r(A) = \inf_{x \in X} r(x, A)$. If there exists a point $c \in X$ such that $r(c, A) = r(A)$, then c is called a *Chebyshev center* of A . Garkavi [4] proved that if X is 1-complemented in X^{**} (in particular, if X is reflexive) then every bounded subset A of X has a Chebyshev center, and if X is uniformly convex in every direction, then the Chebyshev center is unique (see also [3, Ch. 2, Notes and remarks]). Consequently, in uniformly convex spaces, every bounded subset A has a unique Chebyshev center [5, Part 5 §33].

It is possible to characterize inner product spaces among normed linear spaces, using the notion of Chebyshev center [2]. Let c_A denote a Chebyshev center of a nontrivial bounded subset A of a Banach space X . In [2], Baronti and Papini proved the following inequality for any nonempty subset A of a Hilbert space H :

$$r^2(x, A) \geq r^2(A) + \|x - c_A\|^2 \text{ for all } x \in H,$$

in particular,

$$r(x, A) > \|x - c_A\| \text{ for all } x \in H,$$

for any nontrivial bounded subset A of H . It clearly follows from the above inequality that in a Hilbert space H , $c_A \notin \text{Far } A$, where c_A is the unique Chebyshev center of a nontrivial bounded subset A of H .

A Banach space X is said to be *strictly convex* if S_X does not contain nontrivial linear segment i.e., there does not exist $u, v \in S_X$ ($u \neq v$) such that $\{tu + (1 - t)v : t \in [0, 1]\} \subset S_X$. Equivalently, X is strictly convex if every $x \in S_X$ is an extreme point of B_X . One more reformulation: X is strictly convex if and only if for every two points $x, y \in X \setminus \{0\}$ with $x \notin \{ty : t > 0\}$, the *strict triangle inequality* $\|x + y\| < \|x\| + \|y\|$ holds true.

It is clear that if the unit sphere of a Banach space X contains a nontrivial line segment $L = \{tu + (1 - t)v : t \in [0, 1]\}$ (i.e., X is not strictly convex), then all the points of A are of the same distance 1 from the origin, so $A = \text{Far } A$ and in particular, the Chebyshev center $\frac{(u+v)}{2}$ belongs to $\text{Far } A$. This observation motivated Debmalaya Sain to ask in “Research Gate” the following question:

Can a Chebyshev center of a bounded set be a farthest point of the set from a point in a strictly convex Banach space?

This question, which we answer in positive, led to other natural questions and answers, and all these resulted in the article which we are presenting now. We are indebted to the “Research Gate” platform that brought the authors of this paper together.

As we will see in this paper, whether the Chebyshev center of a nontrivial subset of a Banach space may belong to the set, is an important factor in determining the convexity properties of the space. In view of the discussions above, let us introduce the following definitions:

Definition 1.1. A set A in a Banach space X is said to be a *CCF set* (comes from Chebyshev center in $\text{Far } A$) if there is a Chebyshev center of A that belongs to $\text{Far } A$. A is said to be a *CCNF set* (comes from Chebyshev center not in $\text{Far } A$) if it is not a CCF set.

Definition 1.2. A Banach space X is said to be *CCF* if it contains a nontrivial CCF set. X is said to be *CCNF* if it is not CCF, i.e., all nontrivial subsets of X are CCNF.

The main results of the paper deal with the general properties of CCF and CCNF spaces. These results are collected in the next section, ingeniously called “Main results”. At first, in Theorem 2.3, for every Banach space X , we reduce the question whether X is CCNF to the question whether for every $y \in S_X$ and every $r \in (0, 1)$, the Chebyshev radius of the set $B_X \cap B[y, r]$ is strictly smaller than r .

From our earlier discussion, it easily follows that every CCNF space must be strictly convex. In Theorem 2.5, using Theorem 2.3 and a geometric lemma, for two-dimensional spaces we prove the converse result: every two-dimensional strictly convex real Banach space is CCNF. However, the result no longer holds true if the dimension of the space is greater than two. We give examples, in both finite-dimensional (Example 2.8) and infinite-dimensional (Example 2.10) Banach spaces, to illustrate the scenario.

The infinite-dimensional example has an interesting additional property that $r(A) = \frac{1}{2} \text{diam}(A)$. Recall, a set with this property is called *centerable*. Our Theorem 2.11 demonstrates impossibility of such examples in uniformly convex spaces: if A is any nontrivial centerable subset of a uniformly convex Banach space X , then A is CCNF. This result

implies the following characterization of finite-dimensional strictly convex Banach spaces (Theorem 2.12):

A finite-dimensional Banach space X is strictly convex if and only if every nontrivial bounded centerable subset of X is CCNF.

The notion of M -compactness also plays a vital role in the study of farthest points. A sequence $\{a_n\}$ in A is said to be *maximizing* if for some $x \in X$, $\|x - a_n\| \rightarrow r(x, A)$. A subset A of X is said to be *M -compact* if every maximizing sequence in A has a subsequence that converge to an element of A . In this paper, in Theorem 2.15, we prove that in a strictly convex Banach space, every nontrivial, bounded, centerable, M -compact set is CCNF. It is also easy to observe that this property characterizes the strict convexity of a Banach space.

In the last short section, we demonstrate that all L_p spaces, with $p \neq 2$, differ dramatically from the Hilbert space in the sense of the properties that we consider in this paper. Namely, although, as we mentioned before, Hilbert spaces are CCNF, all non-Hilbert L_p spaces of dimension greater than two are CCF.

2. MAIN RESULTS

Our first goal is to obtain a geometric characterization of CCNF Banach spaces. To this end, we first reduce the CCNF property of a Banach space to subsets of the form “intersection of the unit ball with a small ball”. The following two lemmas extract the main ideas of the proof.

Lemma 2.1. *Let A be a nontrivial bounded subset of X , $x \in \text{Far } A$. Then, for every $N > 0$ there is a point $y \in X$ such that x is a farthest point of A from y and $\|x - y\| > N$.*

Proof. According to the definition of $\text{Far } A$, there is a $z \in X$ such that $\|x - z\| \geq \|a - z\|$ for all $a \in A$. Let us demonstrate that for any $t > 1$, x is a farthest point of A from $tz + (1 - t)x$. Indeed, for any $a \in A$,

$$\begin{aligned} \|(tz + (1 - t)x) - a\| &\leq \|tz + (1 - t)x - z\| + \|z - a\| \\ &\leq (t - 1)\|z - x\| + \|z - x\| \\ &= t\|z - x\| = \|(tz + (1 - t)x) - x\|. \end{aligned}$$

We observe that $\|tz + (1 - t)x\| = \|t(z - x) + x\| \geq t\|z - x\| - \|x\| \rightarrow \infty$ as $t \rightarrow \infty$. Consequently, for sufficiently large t , the point $y = tz + (1 - t)x$ is what we are looking for. \square

Lemma 2.2. *Let A be a nontrivial bounded subset of X , containing its Chebyshev center c_A . Suppose c_A is at the same time a farthest point of A from some $y \in X$. Let r be the Chebyshev radius of A and $R = \|c_A - y\|$. Then $r \leq R$ and the subset $U = B[c_A, r] \cap B[y, R]$ has the following properties:*

- (a) $A \subseteq U$.

- (b) *The Chebyshev radius of U equals r .*
- (c) *c_A is a Chebyshev center of U .*
- (d) *c_A is a farthest point of U from y .*

Proof. Inclusions $A \subseteq B[c_A, r]$ and

$$A \subseteq B[y, R] \quad (2.1)$$

follow from definitions of Chebyshev center and of farthest point respectively. Consequently, (a) is correct. Because of (2.1), the Chebyshev radius r of A cannot be greater than R . Property (a) implies $r(U) \geq r$, and inclusion

$$U \subseteq B[c_A, r] \quad (2.2)$$

implies the reverse inequality, which proves (b). Taking (b) into account, we see that (2.2) means (c). Finally, (d) follows from the fact that $c_A \in A \subset U$ and from the inclusion $U \subseteq B[y, R]$. \square

Now we are ready to prove the following characterization of CCNF Banach spaces.

Theorem 2.3. *Denote $r_{t,z}$ the Chebyshev radius of the set $A_{t,z} = B_X \cap B[z, t]$. Then, for a Banach space X the following three conditions are equivalent:*

- (i) *X is a CCNF space;*
- (ii) *for every $z \in S_X$ and every $t \in (0, 1]$, the inequality $r_{t,z} < t$ holds true;*
- (iii) *for every $\varepsilon \in (0, 1]$, there is a $t_0 \in (0, \varepsilon)$ such that for every $z \in S_X$ and every $t \in (0, t_0]$, the inequality $r_{t,z} < t$ holds true.*

Proof. (i) \Rightarrow (ii). As $A_{t,z} \subseteq B[z, t]$ we have $r_{t,z} \leq t$. If $r_{t,z} = t$, then z is a Chebyshev center of $A_{t,z}$. At the same time, z is a farthest point of $A_{t,z}$ from the origin, which contradicts our assumption (i). Consequently, $r_{t,z} < t$.

The implication (ii) \Rightarrow (iii) is evident, so it remains to prove (iii) \Rightarrow (i). Assume contrary that X is CCF. Then, by definition, there exists a nontrivial bounded subset A of X , containing its Chebyshev center c_A , such that $c_A \in \text{Far } A$. Applying Lemma 2.1 for a given $N > 0$, we can find a $y \in X$ such that c_A is a farthest point of A from y and $R := \|c_A - y\| > N$. Denote r the Chebyshev radius of A . According to Lemma 2.2, $r \leq R$. Denote $t = \frac{r}{R} \in (0, 1]$. Consider the set $U = B[c_A, r] \cap B[y, R]$ from Lemma 2.2. According to (b) of that lemma, $r(U) = r$.

For every $x \in X$, denote $f(x) = \frac{1}{R}(x - y)$. Observe that $f(y) = 0$, $\|f(c_A)\| = 1$ and f multiplies all the distances by the same coefficient $\frac{1}{R}$, i.e., $\|f(x_1) - f(x_2)\| = \frac{1}{R}\|x_1 - x_2\|$ for all $x_1, x_2 \in X$. Consequently, $r(f(U)) = \frac{r}{R} = t$. On the other hand,

$$f(U) = B[f(c_A), \frac{r}{R}] \cap B[f(y), 1] = B[f(c_A), t] \cap B[0, 1] = A_{t, f(c_A)}.$$

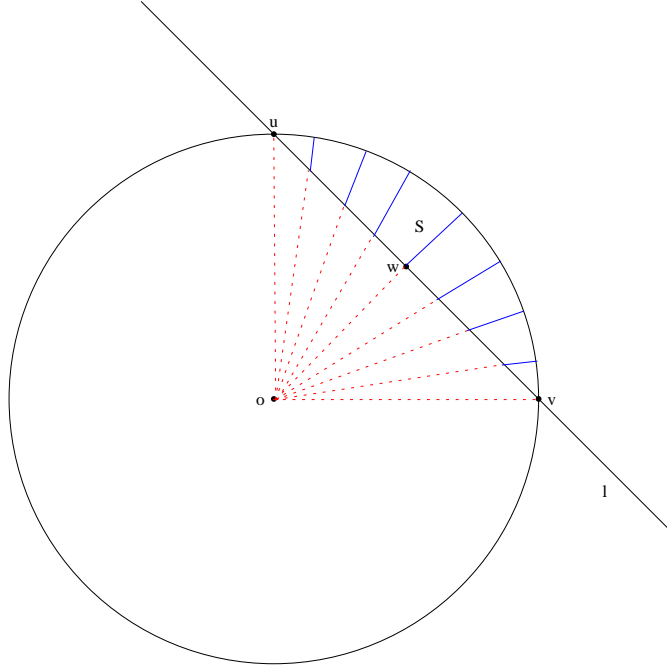
So, $r_{t,z} = t$ for $z = f(c_A) \in S_X$ and $t = \frac{r}{R} \leq \frac{r}{N} \rightarrow 0$ as $N \rightarrow \infty$. This contradicts our assumption (iii). \square

We next prove that in a two-dimensional strictly convex real Banach space X , every nontrivial bounded subset of X is CCNF. To this end, we need the following lemma:

Lemma 2.4. *Let X be a two-dimensional real Banach space, $u, v \in S_X$ and let the straight line l that connects u and v does not contain origin θ . Let S denote the part of B_X not containing θ , that is cut from B_X by l ; $w = \frac{u+v}{2}$, $r = \|u - w\| = \|v - w\| = \frac{1}{2}\|u - v\|$. Then, $S \subset w + rB_X$, i.e., the distance of every point of S to w does not exceed r .*

Proof. Clearly, it is sufficient to prove that $\|s - w\| \leq r$ for all $s \in S$. Let $w_t = (1 - t)u + tw$, $0 \leq t \leq 1$ and $w_{t'} = (1 - t')w + t'v$, $0 \leq t' \leq 1$. Now,

$$\begin{aligned} \|u - w_t\| + \|w_t - w\| &= \|u - (1 - t)u - tw\| + \|(1 - t)u + tw - w\| \\ &= t\|u - w\| + (1 - t)\|u - w\| \\ &= \|u - w\| = r. \end{aligned} \tag{2.3}$$



Similarly,

$$\|w - w_{t'}\| + \|w_{t'} - v\| = \|w - v\| = r. \tag{2.4}$$

Since X is a two-dimensional real Banach space, for any $s \in S$, either $s = \lambda w_t$ or $s = \lambda w_{t'}$, for some $\lambda \geq 1$. We have, $\|\lambda w_t\| \leq 1 \Rightarrow \lambda \leq \frac{1}{\|w_t\|}$

and also, $\|\lambda w_{t'}\| \leq 1 \Rightarrow \lambda \leq \frac{1}{\|w_{t'}\|}$. Now,

$$\begin{aligned} \|\lambda w_t - w_t\| &= (\lambda - 1)\|w_t\| \\ &\leq \left(\frac{1}{\|w_t\|} - 1\right)\|w_t\| \\ &= 1 - \|w_t\| \\ &= \|u\| - \|w_t\| \leq \|u - w_t\|. \end{aligned} \quad (2.5)$$

Similarly,

$$\|\lambda w_{t'} - w_{t'}\| \leq \|v - w_{t'}\|. \quad (2.6)$$

Now using (2.3) and (2.5), we have,

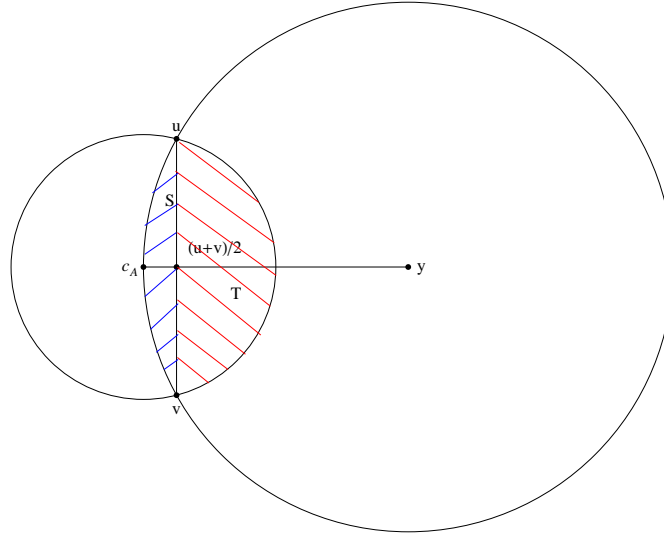
$$\begin{aligned} \|\lambda w_t - w\| &= \|\lambda w_t - w_t + w_t - w\| \\ &\leq \|\lambda w_t - w_t\| + \|w_t - w\| \\ &\leq \|u - w_t\| + \|w_t - w\| \\ &= \|u - w\| = r. \end{aligned}$$

Similarly, using (2.4) and (2.6), we can show that $\|\lambda w_{t'} - w\| \leq r$. So for all $s \in S$, $\|s - w\| \leq r$, which completes the proof. \square

Now we are ready to prove the promised theorem.

Theorem 2.5. *Let X be a two-dimensional strictly convex real Banach space. Let A be a nontrivial bounded subset of X , containing its Chebyshev center c_A . Then A is CCNF.*

Proof. We will use the notations of Lemma 2.2.



Suppose $c_A \in A$ is a farthest point of A from some $y \in X$. Let r be the Chebyshev radius of A and $R = \|x - y\|$. Let u, v be the intersection points of the spheres $S[c_A, r]$ and $S[y, R]$. Then by Lemma 2.4, both

S and T in the above picture are subsets of the closed ball centered at $\frac{(u+v)}{2}$ and radius $\|\frac{(u-v)}{2}\|$. Then $A \subseteq B[\frac{(u+v)}{2}, \|\frac{(u-v)}{2}\|]$. By the definition of Chebyshev radius, $\|\frac{(u-v)}{2}\| \geq r$ which implies that $\|u - v\| \geq 2r$. On the other hand, $u, v \in S[c_A, r]$, so $\|u - v\| \leq 2r$ and consequently $\|u - v\| = 2r$. We have $\|(u - c_A) + (c_A - v)\| = 2r$ and $\|u - c_A\| + \|c_A - v\| = 2r$. As the space is strictly convex, we must have $(u - c_A) = k(c_A - v)$, for some constant $k > 0$. Since $\|u - c_A\| = \|c_A - v\| = r$, we have $k = 1$. Therefore, we have $c_A = \frac{u+v}{2}$. Now $u, v, c_A \in S[y, R]$ and so by strict convexity we get $u = v = c_A$. Then $r = 0$ and so A consists of only one point, contradicting our assumption that A is nontrivial. This completes the proof of the theorem. \square

The converse of Theorem 2.5 is also true. Indeed, as we already remarked in the introduction, if X is not strictly convex, then S_X contains a straight line segment $L = \{(1 - t)u + tv : u, v \in S_X, t \in [0, 1]\}$. It is easy to see that $\frac{u+v}{2}$ is a Chebyshev center of L , which is also a farthest point of L from the origin. Thus, we have the following characterization of strict convexity of a two-dimensional real Banach space:

Theorem 2.6. *A two-dimensional real Banach space X is strictly convex if and only if every nontrivial bounded subset A which contains its Chebyshev center is CCNF.*

In general Theorem 2.5 is not true if the dimension of the space is strictly greater than two. The following two examples illustrate the situation in both finite and infinite-dimensional strictly convex spaces. Firstly we recall an easy but useful way to construct equivalent strictly convex norms [3, Ch. 4 §2, Theorem 1].

Proposition 2.7. *Let X, Y be Banach spaces, Y be strictly convex and let $T: X \rightarrow Y$ be an injective continuous linear operator. For $x \in X$, denote $p(x) = \|x\| + \|Tx\|$. Then (X, p) is strictly convex.*

Example 2.8. Consider $X = (\mathbb{R}^n, \|\cdot\|)$, $n \geq 3$ where

$$\|(x_1, x_2, \dots, x_n)\| = \sum_{i=1}^n |x_i| + \frac{1}{2} \sqrt{\sum_{i=1}^n |x_i|^2}.$$

It is easy to see that the norm is of the form given by Proposition 2.7. So X is strictly convex and by finite-dimensionality, X is uniformly convex as well. Consequently, for any bounded set, the Chebyshev center is unique.

Let $\{e_1, \dots, e_n\}$ be the canonical basis of \mathbb{R}^n , i.e., $e_1 = (1, 0, 0, \dots, 0)$, $e_2 = (0, 1, 0, 0, \dots, 0)$, etc. Also denote $\theta = (0, 0, \dots, 0)$. Let

$$A = \{\theta, e_1, e_2, \dots, e_n\}.$$

Consider $z = (1, 1, \dots, 1) \in \mathbb{R}^n$. Then $\|z - e_k\| = (n-1) + \frac{\sqrt{(n-1)}}{2}$ for all $k = 1, \dots, n$. However,

$$\|z - \theta\| = n + \frac{\sqrt{n}}{2} > (n-1) + \frac{\sqrt{(n-1)}}{2},$$

which proves that θ is the farthest point of A from z .

We claim that θ is the Chebyshev center of A . If $(a_1, a_2, \dots, a_n) \in \mathbb{R}^n$ is a Chebyshev center of A , then by symmetry, all the cyclic permutations of coordinates

$$(a_2, a_3, \dots, a_n, a_1), \dots, (a_n, a_1, \dots, a_{n-1})$$

give us Chebyshev centers of A as well. Since the set of all Chebyshev centers is convex, $(\alpha, \alpha, \dots, \alpha)$ is also a Chebyshev center of A , where $\alpha = \frac{a_1 + a_2 + \dots + a_n}{n}$. By uniqueness of Chebyshev center in uniformly convex spaces, the Chebyshev center of A should be of the form sz , $s \in \mathbb{R}$. As $\|a - \theta\| = \frac{3}{2}$ for all $a \in A \setminus \{\theta\}$, it is sufficient to demonstrate that for any $s \in \mathbb{R}$, there is a $p \in A$ such that $\|sz - p\| \geq \frac{3}{2}$.

If $s \geq 1$ then considering $p = \theta$ we are done. If $s < 0$ then considering e_1 as p we are also done. Let $0 < s < 1$. In this case, let us also take $p = e_1$. We have,

$$\begin{aligned} 2s > s &\Rightarrow (1 - 2s) < (1 - s) \\ &\Rightarrow 1 - (2s + 2s - 2s) < (1 - s) \leq \sqrt{(1 - s)^2 + s^2 + \dots + s^2} \\ &\Rightarrow (s + s - s) + \frac{1}{2}\sqrt{(1 - s)^2 + s^2 + \dots + s^2} > \frac{1}{2} \\ &\Rightarrow (1 - s) + s + \dots + s + \frac{1}{2}\sqrt{(1 - s)^2 + s^2 + \dots + s^2} > \frac{3}{2} \\ &\Rightarrow \|sz - e_1\| > \frac{3}{2}. \end{aligned}$$

This proves that θ is the Chebyshev center of A .

Remark 2.9. Applying Lemma 2.2 to the set A , from Example 2.8, we deduce that there exists a finite-dimensional uniformly convex Banach space X and a non-trivial convex compact set $U \subset X$ such that U is a CCF subset of X .

Now we present a similar example with a centerable subset. The example “lives” in an infinite-dimensional strictly convex Banach space X . Afterwards, it will follow from Theorem 2.11 that such an example is impossible in finite-dimensional strictly convex Banach spaces.

Example 2.10. Consider the space c_0 of all sequences of real numbers converging to zero, equipped with the following norm:

$$\|x\| = \max_k |x_k| + \sqrt{\sum_{k=1}^{\infty} \frac{1}{4^k} |x_k|^2} \quad (2.7)$$

where x_k ($k \in \mathbb{N}$) denote the k -th coordinate of $x \in c_0$. Clearly, the norm is strictly convex. Let us denote this Banach space by X . Let $\theta = (0, 0, \dots, 0, \dots)$ and $e_n = (0, 0, \dots, 0, 1, 0, \dots)$, i.e., the n -th coordinate of e_n is 1 and all other coordinates are 0. Denote

$$x_n = \frac{1}{n}e_1 + \left(1 - \frac{1}{n}\right)e_n, \quad y_n = \frac{1}{n}e_1 - \left(1 - \frac{1}{n}\right)e_n$$

and consider $A = \{\theta\} \cup \{x_n : n = 2, 3, \dots\} \cup \{y_n : n = 2, 3, \dots\}$.

We claim that A is a subset of the unit ball and consequently, $r(A) \leq 1$. In fact,

$$\|x_n\| = \|y_n\| = \left(1 - \frac{1}{n}\right) + \sqrt{\frac{1}{4n^2} + \frac{1}{4n} \left(1 - \frac{1}{n}\right)^2}.$$

Since $\frac{1}{4n} \leq \frac{1}{4n^2}$ for all $n = 2, 3, 4, \dots$,

$$\|x_n\| = \|y_n\| \leq \left(1 - \frac{1}{n}\right) + \sqrt{\frac{2}{4n^2}} < 1 - \frac{1}{n} + \frac{1}{n} = 1.$$

The claim is proved. Now,

$$\lim_{n \rightarrow \infty} \|x_n - y_n\| = \lim_{n \rightarrow \infty} 2 \left(1 - \frac{1}{n}\right) \|e_n\| \geq \lim_{n \rightarrow \infty} 2 \left(1 - \frac{1}{n}\right) = 2.$$

Consequently, $\text{diam}(A) \geq 2$. Since $r(A) \geq \frac{1}{2} \text{diam}(A)$, we have $r(A) = 1$. So θ is a Chebyshev center of A . Finally we prove that θ is a farthest point of A from $u = e_1$. In fact, $\|e_1 - \theta\| = \|e_1\| = \frac{3}{2}$. On the other hand,

$$\begin{aligned} \|e_1 - x_n\| &= \|e_1 - y_n\| = \left(1 - \frac{1}{n}\right) \|e_1 \pm e_n\| \\ &= \left(1 - \frac{1}{n}\right) \left(1 + \sqrt{\frac{1}{4} + \frac{1}{4n}}\right) \\ &\leq \left(1 - \frac{1}{n}\right) \left(\frac{3}{2} + \frac{1}{2n}\right) \\ &= \frac{3}{2} + \frac{1}{2n} - \frac{3}{2n} - \frac{1}{n2^n} < \frac{3}{2}. \end{aligned}$$

So θ is the farthest point of A from e_1 .

Next, we prove that if A is a bounded centerable subset in a uniformly convex Banach space, then A is CCNF. Before doing this, let us recall one of the standard equivalent definitions of uniform convexity: a Banach space X is said to be *uniformly convex* if for every two sequences $\{x_n\}, \{y_n\}$ in B_X , the condition $\lim_{n \rightarrow \infty} \|x_n + y_n\| = 2$ implies $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

Theorem 2.11. *Let X be a uniformly convex Banach space and A be a nontrivial bounded centerable subset of X , containing its Chebyshev center c_A . Then A is CCNF.*

Proof. Let $r > 0$ be the Chebyshev radius of A . According to the definition of a centerable set, there are $u_n, v_n \in A$, $n = 1, 2, \dots$ such that

$$\lim_{n \rightarrow \infty} \|u_n - v_n\| = 2r. \quad (2.8)$$

Consider elements

$$x_n = \frac{1}{r}(u_n - c_A), \quad y_n = \frac{1}{r}(c_A - v_n).$$

Then $x_n, y_n \in B_X$, $\lim_{n \rightarrow \infty} \|x_n + y_n\| = 2$, so the uniform convexity of X implies $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$. This means that $\lim_{n \rightarrow \infty} \|u_n + v_n - 2c_A\| = 0$. In other words,

$$u_n + v_n \rightarrow 2c_A.$$

Suppose c_A is a farthest point of A from some $y \in X$. Denote $R = \|c_A - y\|$. Now, denote

$$\tilde{x}_n = \frac{1}{R}(u_n - y), \quad \tilde{y}_n = \frac{1}{R}(v_n - y).$$

Then $\tilde{x}_n, \tilde{y}_n \in B_X$,

$$\lim_{n \rightarrow \infty} \|\tilde{x}_n + \tilde{y}_n\| = \frac{1}{R} \lim_{n \rightarrow \infty} \|u_n + v_n - 2y\| = \frac{1}{R} \|2c_A - 2y\| = 2.$$

Again, the uniform convexity of X implies $\lim_{n \rightarrow \infty} \|\tilde{x}_n - \tilde{y}_n\| = 0$, i.e., $\|u_n - v_n\| \rightarrow 0$, which contradicts (2.8). This contradiction completes the proof of the theorem. \square

Since in the finite-dimensional case, strict convexity implies uniform convexity, it is possible to obtain the following characterization of finite-dimensional strictly convex Banach spaces, simply by observing that any straight line segment in a Banach space is always a centerable set.

Theorem 2.12. *A finite-dimensional Banach space X is strictly convex if and only if every nontrivial bounded centerable subset A which contains its Chebyshev center is CCNF.*

Remark 2.13. Example 2.10 shows that the uniform convexity condition in Theorem 2.11 cannot be substituted by strict convexity.

In the next theorem, we prove that if A is a bounded centerable M-compact subset in a strictly convex Banach space, then A is CCNF. Before proving the theorem we first prove the following lemma:

Lemma 2.14. *Let X be a Banach space. Let A be any nontrivial bounded centerable M-compact subset of X , containing its Chebyshev center c_A . Then A attains its diameter.*

Proof. Since in our case,

$$\text{diam}(A) = \sup_{a,b \in A} \|a - b\| = 2r(A),$$

there exist sequences $\{x_n\}, \{y_n\} \subset A$ such that $\|x_n - y_n\| \rightarrow 2r(A)$. We claim that $\{x_n\}$ is a maximizing sequence in A for c_A . If not, then there exists $\varepsilon_0 > 0$ and a subsequence $\{x_{n_k}\}$ such that $\|c_A - x_{n_k}\| \leq r(A) - \varepsilon_0$. Then,

$$\begin{aligned} \|x_{n_k} - y_{n_k}\| &= \|(x_{n_k} - c_A) + (c_A - y_{n_k})\| \\ &\leq \|x_{n_k} - c_A\| + \|c_A - y_{n_k}\| \\ &\leq r(A) - \varepsilon_0 + r(A) = 2r(A) - \varepsilon_0, \end{aligned}$$

which contradicts the fact that $\|x_n - y_n\| \rightarrow 2r(A)$. By the same argument, $\{y_n\}$ is a maximizing sequence in A for c_A . Consequently, as A is M-compact, there is a subsequence $\{n_k\} \subset \mathbb{N}$ and there are $\tilde{x}, \tilde{y} \in A$ such that $x_{n_k} \rightarrow \tilde{x}$ and $y_{n_k} \rightarrow \tilde{y}$. Then

$$\text{diam}(A) = \lim_{n \rightarrow \infty} \|x_n - y_n\| = \lim_{k \rightarrow \infty} \|x_{n_k} - y_{n_k}\| = \|\tilde{x} - \tilde{y}\|.$$

Thus diameter of A is attained. \square

We now prove the desired theorem.

Theorem 2.15. *Let X be a strictly convex Banach space and $A \subset X$ be a nontrivial bounded centerable M-compact subset, containing its Chebyshev center c_A . Then A is CCNF.*

Proof. Suppose A is CCF. Then $c_A \in \text{Far } A$. By the definition, there exists $x \in X$ such that $c_A \in F(x, A)$. Denote

$$R = \|x - c_A\| = \sup_{a \in A} \|x - a\|.$$

Due to Lemma 2.14, $\text{diam}(A)$ is attained and since A is centerable, $\text{diam}(A) = 2r(A)$. This means that there exist $a_1, a_2 \in A$ such that

$$\|a_1 - a_2\| = \sup_{a,b \in A} \|a - b\| = 2r(A). \quad (2.9)$$

We claim that $\|c_A - a_1\| = \|c_A - a_2\| = r(A)$. Clearly $\|c_A - a_1\| \leq r(A)$ and $\|c_A - a_2\| \leq r(A)$. Moreover, the assumption that one of them is strictly smaller than $r(A)$ leads to a contradiction:

$$2r(A) = \|a_1 - a_2\| = \|a_1 - c_A + c_A - a_2\| \leq \|a_1 - c_A\| + \|a_2 - c_A\| < 2r(A).$$

So, the claim is proved. Now,

$$\left\| \frac{1}{2} ((a_1 - c_A) + (c_A - a_2)) \right\| = r(A).$$

Geometrically this means that $a_1 - c_A, c_A - a_2$ and $\frac{1}{2}((a_1 - c_A) + (c_A - a_2))$ belong to the same sphere $r(A)S_X$. By the strict convexity of X , it follows that $a_1 - c_A = c_A - a_2$, i.e., $c_A = \frac{1}{2}(a_1 + a_2)$. The following chain

of inequalities

$$\begin{aligned} R &= \|x - c_A\| = \left\| \frac{1}{2}((x - a_1) + (x - a_2)) \right\| \\ &\leq \frac{1}{2}\|x - a_1\| + \frac{1}{2}\|x - a_2\| \leq \sup_{a \in A} \|x - a\| = R \end{aligned}$$

implies that all of them are equalities, i.e., all three vectors $x - a_1$, $x - a_2$, and $\frac{1}{2}((x - a_1) + (x - a_2))$ belong to the same sphere RS_X . Then, the strict convexity of X implies that $x - a_1 = x - a_2$, i.e., $a_1 = a_2$. This contradiction with (2.9) completes the proof of the theorem. \square

Remark 2.16. Example 2.10 shows that the M-compactness condition in Theorem 2.15 can not be removed.

Now, we can give an characterization of strictly convex Banach spaces, simply by observing that any closed straight line segment in a Banach space is always a centerable and M-compact set. Thus, we have the following theorem :

Theorem 2.17. *A Banach space X is strictly convex if and only if every nontrivial bounded centerable and M-compact subset $A \subset X$ which contains its Chebyshev center is CCNF.*

Remark 2.18. Theorem 2.15 shows that the uniform convexity condition in Theorem 2.11 can be substituted by strict convexity if we impose an additional condition of M-compactness on the subset A of X .

We would like to add a final comment that Theorem 2.6, Theorem 2.12 and Theorem 2.17 together yield a nice step by step characterization of strict convexity of a Banach space. The characterizing properties follow an interesting trend, depending on the dimension of the space. Accordingly, we state the following theorem as the final result of this section:

Theorem 2.19. *Let X be a Banach space. Then the following holds.*

- (a) *If X is a two-dimensional real Banach space, then X is strictly convex if and only if every nontrivial bounded subset A which contains its Chebyshev center is CCNF.*
- (b) *If X is a finite-dimensional Banach space, then X is strictly convex if and only if every nontrivial bounded centerable subset A which contains its Chebyshev center is CCNF.*
- (c) *If X is any Banach space, then X is strictly convex if and only if every nontrivial bounded centerable and M-compact subset A which contains its Chebyshev center is CCNF.*

3. CHEBYSHEV CENTERS IN L_p SPACES

In this section we demonstrate that all Banach spaces L_p , $p \neq 2$, of dimension greater than two are CCF. Since L_1 and L_∞ are not strictly

convex, for them this result follows from the previous discussion. So, in this section we consider only $1 < p < \infty$. We begin with an elementary technical proposition in dimension three. Let $\ell_p^{(3)}$ denote the space \mathbb{R}^3 equipped with the norm $\|(x_1, x_2, x_3)\| = (|x_1|^p + |x_2|^p + |x_3|^p)^{1/p}$, and let $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$.

Proposition 3.1. *The Chebyshev center of the set $A_0 = \{e_1, e_2, e_3\} \subset \ell_p^{(3)}$ is the point $x_p = (s_p, s_p, s_p)$, where*

$$s_p = \frac{1}{1 + 2^{1/(p-1)}}.$$

Proof. Since $\ell_p^{(3)}$ is uniformly convex, A_0 possesses unique Chebyshev center and by symmetry, this Chebyshev center must be of the form (s, s, s) . What remains to do, is to minimize the quantity

$$f(s) = \|e_k - (s, s, s)\|^p = |1 - s|^p + 2|s|^p, s \in \mathbb{R}.$$

Evidently, the minimum attains on $(0, 1)$ (otherwise $f(s) \geq 1$), where $f'(s) = 2ps^{p-1} - p(1 - s)^{p-1}$, and s_p is the unique root of equation $f'(s) = 0$. \square

Following the notation of the previous proposition, denote $A_p = \{e_1, e_2, e_3, x_p\} \subset \ell_p^{(3)}$.

Proposition 3.2. *For $p \in (1, 2) \cup (2, \infty)$, A_p is a CCF set and consequently, $\ell_p^{(3)}$ is a CCF space.*

Proof. A_p is formed by A_0 together with its Chebyshev center x_p , so x_p is also the Chebyshev center of A_p . It remains to show that $x_p \in \text{Far } A_p$. We consider the following two cases separately:

Case 1: $p \in (1, 2)$. In this case

$$0 < s_p < \frac{1}{3}. \quad (3.1)$$

We are going to demonstrate that for $t > 1$ large enough, x_p is the farthest point of A_p from $y = (t, t, t)$. The distance from y to any of e_k equals $((t - 1)^p + 2t^p)^{1/p}$, $\|y - x_p\| = 3^{1/p}(t - s_p)$, so we need to check for large t the inequality

$$(t - 1)^p + 2t^p < 3(t - s_p)^p.$$

Dividing by t^p and denoting $\tau = \frac{1}{t}$, we reduce this to

$$(1 - \tau)^p + 2 < 3(1 - s_p\tau)^p \quad (3.2)$$

for small positive τ . At the point $\tau = 0$, the left-hand side of (3.2) equals the right-hand side. So in order to demonstrate (3.2) for τ close to 0, it is sufficient to show for $f_1(\tau) = (1 - \tau)^p + 2$, $f_2(\tau) = 3(1 - s_p\tau)^p$, the validity of the inequality $f_1'(0) < f_2'(0)$. This is the inequality

$$-p < -3ps_p,$$

which follows from (3.1).

Case 2: $p \in (2, \infty)$. In this case

$$s_p > \frac{1}{3}. \quad (3.3)$$

We are going to demonstrate that for $t > 0$ large enough, x_p is the farthest point of A_p from $y = (-t, -t, -t)$. The distance from y to any of e_k equals $((t+1)^p + 2t^p)^{1/p}$, $\|y - x_p\| = 3^{1/p}(t + s_p)$, so we need to check for large t the inequality

$$(t+1)^p + 2t^p < 3(t + s_p)^p.$$

The same way as above, this reduces to

$$(1 + \tau)^p + 2 < 3(1 + s_p \tau)^p$$

for small positive τ . Denoting $g_1(\tau) = (1 + \tau)^p + 2$, $g_2(\tau) = 3(1 + s_p \tau)^p$, we have to demonstrate the inequality $g_1'(0) < g_2'(0)$, i.e., the inequality

$$p < 3ps_p,$$

which follows from (3.3). \square

Theorem 3.3. *Let (Ω, Σ, μ) be a finite or σ -finite measure space, containing a disjoint triple $\{\Delta_i\}_{i=1}^3 \subset \Sigma$ of subsets of finite positive measure. Then $L_p = L_p(\Omega, \Sigma, \mu)$ is a CCF space for every $p \in (1, 2) \cup (2, \infty)$.*

Proof. Denote $f_i = \mathbf{1}_{\Delta_i} / \|\mathbf{1}_{\Delta_i}\|$, $i = 1, 2, 3$, $E = \text{lin}\{f_i\}_{i=1}^3 \subset L_p$. It is well-known (and can be checked easily) that E is isometric to $\ell_p^{(3)}$, where the corresponding isometry $T : \ell_p^{(3)} \rightarrow E$ acts as follows:

$$T(x_1, x_2, x_3) = x_1 f_1 + x_2 f_2 + x_3 f_3.$$

It is also well-known that E is 1-complemented in L_p with the corresponding projection $P : L_p \rightarrow E$ being

$$Pf = \sum_{i=1}^3 \frac{\int_{\Delta_i} f d\mu}{\mu(\Delta_i)} \mathbf{1}_{\Delta_i}.$$

(Equality $\|P\| = 1$ follows from Hölder's inequality).

Let A_p be the set from Proposition 3.2. If we consider $T(A_p)$ as a subset of E , then Tx_p is its Chebyshev center, because T is an isometry. Since E is 1-complemented in L_p , Tx_p is also the Chebyshev center of $T(A_p)$, when $T(A_p)$ is considered as a subset of L_p . Let $y \in \ell_p^{(3)}$ be such a point that $x_p \in F(y, A_p)$. Since T is an isometry, Tx_p is the farthest point in $T(A_p)$ from Ty . This means that the Chebyshev center Tx_p of $T(A_p) \subset L_p$ is a farthest point. \square

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